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INTERIOR ALGEBRAS

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This paper presents results previously published in "Naturman, Colin Ashley. Interior algebras and topology. Diss. University of Cape Town, 1991, Chap 3 Neighbourhoods, Convergence and Accumulation §3.1-3.4"

Core material from this paper was also subsequently published with new exposition in the journal article "Naturman, Colin. Countability Properties in Interior Algebras. Quaestiones Mathematicae 17.2 (1994): 189-204."

**R E S E A R C H R E P O R T S
D E P A R T M E N T O F M A T H E M A T I C S
T H E U N I V E R S I T Y O F C A P E T O W N**

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CONVERGENCE AND ACCUMULATION IN INTERIOR ALGEBRAS

Colin Naturman

Abstract: Interior algebras are Boolean algebras enriched with an interior operator and a corresponding closure operator. Alternative descriptions of interior algebras in terms of generalized topologies in Boolean algebras and neighbourhood functions on Boolean algebras are found. The topological concepts of convergence and accumulation of systems and nets are generalized to interior algebras. Relationships between different forms of convergence and accumulation are found.

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Introduction

An interior algebra is a Boolean algebra enriched with an interior operator and a corresponding closure operator. (See the definition below for details). Interior algebras were introduced by McKinsey and Tarski in [3] as an algebraic generalization of topological spaces. Besides their connection to topology, it is well known that interior algebras play the same role for the modal logic $S4$ as Boolean algebras play for ordinary propositional logic. Interior algebras have appeared in the literature in slightly different forms under various names such as *closure algebras* (the original name used for them by McKinsey and Tarski), $S4$ -algebras, *topo-Boolean algebras* and *topological Boolean algebras* (not to be confused with the topological Boolean algebras of topological algebra). The term *interior algebra* has become the standard name for these algebras in English language publications.

Most of the research on interior algebras has focussed on the application of interior algebras to modal logic. There has been little research on interior algebras for their own sake, although many purely interior algebraic results have been produced by modal logicians, especially in connection with varieties and quasi-varieties of interior algebras which are related to extensions of the modal logic $S4$. (See [2] for an introduction to modal logic). In this paper and in subsequent papers we will investigate interior algebras, not in the direction of modal logic, but in the direction of their relationship to topology. Although the results obtained are inspired by topology it will be noticed that they are in fact purely algebraic. Our main aim is to develop the theory of interior algebras further for their own sake, not merely as an aid to topology or modal logic.

In §0 we summarize the preliminaries needed for this paper. In §1 we introduce the concepts of generalized topological spaces and neighbourhood functions (Definitions 1.1 and 1.8) which provide alternative descriptions of interior algebras (Theorems 1.4 and 1.11) and facilitate the generalization of topological concepts to interior algebras. In §2 we consider the convergence of subsets of interior algebras (Definitions 2.1) which is a generalization of system convergence in topology. We introduce the concept of an encloser (Definition 2.6) and use this to generalize system accumulation in topology to subset accumulation in interior algebras (Definition 2.10). Connections between subset convergence and subset accumulation are established by means of a section operator (Definition 2.12, Theorem 2.16, Propositions 2.17 and 2.19, Corollary 2.18). The main purpose of §3 is to provide a general background to the theory of nets and sequences in interior algebras. We develop results concerning nets and sequences in interior algebras and their connection to filters and filter bases (Theorems 3.4, 3.7, 3.15 and 3.17). Most of the results in this section only make

use of the Boolean algebraic aspects of interior algebras and are thus actually results about Boolean algebras. No generality is lost by stating these results for interior algebras since every Boolean algebra can be turned into an interior algebra. §4 deals with the convergence and accumulation of nets and sequences in interior algebras (Definitions 4.1 and 4.8) which are generalizations of the convergence and accumulation of nets and sequences in topology. Relationships between net and sequence convergence and subset convergence are found (Propositions 4.4 and 4.5, Remark 4.6 and Theorems 4.7 and) as are connections between net convergence and net accumulation and sequence convergence and sequence accumulation (Theorems 4.14 and 4.18). As corollaries we obtain connections between subset convergence and net and sequence accumulation (Corollaries 4.15 and 4.20). Net accumulation and subset accumulation are generally very different but we still manage to obtain a connection in certain cases (Propositions 4.11 and 4.12).

Note on Notation

Bold capitals will be used to denote structures. If $A, B, C, \dots$ are structures, the corresponding normal capitals $A, B, C, \dots$ will denote the underlying sets of these structures. If A and B are structures the $A \cong B$ denotes that A is isomorphic to B . If X is a set then, as usual, $\mathcal{P}(X)$ will denote the power set of X . Further notation will be explained as needed.

0. Preliminaries

Basic results concerning interior algebras can be found in [1], [3] and [5]. We revise some of the more important definitions and results below.

An *interior algebra* is an algebraic structure $\langle B, ^I, ^C \rangle$ where:

- (i) $B = \langle B, \cdot, +, ', 0, 1 \rangle$ is a Boolean algebra
- (ii) ^{I} and ^{C} are unary operations satisfying:

- (a) $1^I = 1$
- (b) $a^I \leq a$
- (c) $a^{II} = a^I$
- (d) $(ab)^I = a^I b^I$
- (e) $a^C = a'^I$ for all $a, b \in B$

The operations ^{I} and ^{C} are called the *interior* and *closure* operators respectively. For $a \in B$, a^I and a^C are known as the *interior* and *closure* of a respectively. The canonical examples of interior algebras are those obtained from topological spaces: if $\langle X, \mathcal{T} \rangle$ is topological space, then the power set Boolean algebra on X , together with the usual topological interior and

closure operators, forms an interior algebra.

Note that the interior and closure operators are always monotone, that is, if $a \leq b$ in an interior algebra, then $a^I \leq b^I$ and $a^C \leq b^C$.

The principle of duality for Boolean algebras can be generalized to interior algebras as follows: Given a sentence φ in a formal language for interior algebras, define the dual of φ to be the sentence φ' obtained from φ by interchanging $\cdot$ and $+$, 0 and 1 , and ^{I} and ^{C} . Then φ holds if and only if φ' holds.

Given an interior algebra A , an element $a \in A$ is said to be *open* if and only if $a^I = a$. The dual notion to that of an open element is that of a *closed* element, that is, $a \in A$ is closed if and only if $a^C = a$. For all $a \in A$, a^I is open and a^C is closed. Note also that $a \in A$ is open if and only if a' is closed.

For an interior algebra A , we denote the set of open elements of A by A^O and the set of closed elements by A^C . Note that $0, 1 \in A^O$ and A^O is closed under finite meets and joins. Therefore $\langle A^O, \cdot, +, 0, 1 \rangle$ is a distributive 0,1-lattice. Moreover, for all $a, b \in A^O$, $a \rightarrow b := (a' + b)^I$ is the relative pseudocomplement of a with respect to b in this 0,1-lattice. Thus the structure $A^O = \langle A^O, \cdot, +, \rightarrow, 0, 1 \rangle$ is a Heyting algebra. Dually we have a Brouwerian algebra (dual Heyting algebra), $A^C = \langle A^C, \cdot, +, *, 0, 1 \rangle$, where for all $a, b \in A^C$, $a * b := (a' b)^C$ is the dual relative pseudocomplement of a with respect to b .

If A is an interior algebra and $a \in A$ we let $[a]$ denote the principal ideal of A generated by a , that is, $[a] = \{b \in A : b \leq a\}$. We can turn $[a]$ into an interior algebra $[a] = \langle [a], \cdot, +, ', 0, a, {}^I_a, {}^C_a \rangle$, where for all $b \in [a]$ we have $b'^a = ab'$, $b^{I_a} = a \cdot (a' + b)^I$, and $b^{C_a} = ab^C$. (Basic results concerning these ideal algebras can be found in [4].)

A subset S of an interior algebra is said to be *open* if and only if $a^I \in S$ for all $a \in S$. Dually, S is *closed* if and only if $a^C \in S$ for all $a \in S$. Open filters are important since they determine the congruences on an interior algebra: If A is an interior algebra and F is an open filter in A then F determines the congruence θ given by $a \theta b$ if and only if $ac = bc$ for some $c \in F$. θ is uniquely determined by F since $1/\theta = F$.

1. Generalized Topological Spaces and Neighbourhood Functions

Definition 1.1: By a *generalized topological space* we mean a structure $\langle B, G \rangle$ where:

- (i) $B = \langle B, \cdot, +, ', 0, 1 \rangle$ is a Boolean algebra
- (ii) G is a unary relation (subset of B) satisfying:
 - (a) $0, 1 \in G$
 - (b) G is closed under arbitrary joins.
 - (c) G is closed under finite meets.
 - (d) For all $a \in B$, $\Sigma \{b \in G : b \leq a\}$ exists.

G is said to be a *generalized topology* in the Boolean algebra B . (If B is complete, then condition (d) holds automatically.)

The above definition of a generalized topology is in a form which emphasizes the connection with topology: If T is a topology on a set X , then T is a generalized topology in the power set Boolean algebra on X . The conditions can be simplified as the following proposition shows.

Proposition 1.2: Let B be a Boolean algebra. A subset $G \subseteq B$ is a generalized topology in B if and only if:

- (a) $1 \in G$
- (b) G is closed under binary meets.
- (c) For all $a \in B$, $\max \{b \in G : b \leq a\}$ exists.

Proof: If G is a generalized topology in B then (a) and (b) hold by definition. For (c): Let $a \in B$. Then $\Sigma \{b \in G : b \leq a\}$ exists and is in G since it is a join of elements of G . It follows that $\Sigma \{b \in G : b \leq a\} = \max \{b \in G : b \leq a\}$. Conversely, suppose that (a), (b), (c) all hold. Then $\max \{b \in G : b \leq 0\}$ exists and must be 0 whence $0 \in G$. (a) and (b) ensure that G is closed under finite meets. It remains to show that G is closed under arbitrary joins. So let $S \subseteq G$ such that ΣS exists. Put $d = \max \{b \in G : b \leq \Sigma S\}$. Then for all $a \in S$, $a \leq \Sigma S$ whence $a \leq d$. Thus $\Sigma S \leq d$. But $d \leq \Sigma S$ and so $\Sigma S = d \in G$ as required. $\square$

Remark 1.3: Generalized topological spaces form a finitely axiomatizable elementary subclass of the class of all Boolean algebras with added unary relation. Let φ be the conjunction of the following sentences in the first order language for Boolean algebras with added unary relation:

- $G(1)$
- $(\forall x)(\forall y)(G(x) \wedge G(y) \Rightarrow G(xy))$
- $(\forall x)(\exists y)(G(y) \wedge y \leq x \wedge (\forall z)(G(z) \wedge z \leq x \Rightarrow z \leq y))$

Then by Proposition 1.2 we see that $\langle B, G \rangle$ is a generalized topological space if and only if $\langle B, G \rangle \models \varphi$. $\square$

We now describe the connection between interior algebras and generalized topological spaces: Given an interior algebra A let A^u denote the underlying Boolean algebra of A . Put $Gt A = \langle A^u, A^0 \rangle$. Given a generalized topological space $\langle B, G \rangle$, define operations ^{I} and ^{C} on B by $a^I = \max \{b \in G : b \leq a\}$ and $a^C = a^{I'}$ or equivalently $a^C = \min \{b' : b \in G \text{ and } a \leq b'\}$, for all $a \in B$. Put $Alg \langle B, G \rangle = \langle B, ^I, ^C \rangle$.

Theorem 1.4: Let A be an interior algebra and let $T = \langle B, G \rangle$ be a generalized topological space. Then:

- (i) $Gt A$ is a generalized topological space.
- (ii) $Alg T$ is an interior algebra.
- (iii) $Gt Alg T = T$
- (iv) $Alg Gt A = A$

Proof: (i): We use Proposition 1.2. We know that $1 \in A^0$ and that A^0 is closed under binary meets. If $a \in A$ then $a^I \in A^0$ with $a^I \leq a$ and for all $b \in A^0$ with $b \leq a$ we have $b = b^I \leq a^I$. Thus $a^I = \max \{b \in A^0 : b \leq a\}$. (ii): $1 \in G$ and so clearly $1 = \max \{b \in G : b \leq 1\} = 1^I$. If $a \in B$ then $a^I = \max \{b \in G : b \leq a\} \leq a$. If $a \in B$ then $a^I \in G$ and so obviously $a^I = \max \{b \in G : b \leq a^I\} = a^{II}$. If $a, b \in B$ then $a^I \in G$ and $b^I \in G$ whence $a^I b^I \in G$. Moreover, $a^I \leq a$ and $b^I \leq b$ whence $a^I b^I \leq ab$ and so $a^I b^I \leq \max \{c \in G : c \leq ab\} = (ab)^I$. On the other hand $(ab)^I \in G$ and $(ab)^I \leq ab \leq a$ whence $(ab)^I \leq \max \{c \in G : c \leq a\} = a^I$. Similarly $(ab)^I \leq b^I$ and so $(ab)^I \leq a^I b^I$ whence in fact $(ab)^I = a^I b^I$. By definition, if $a \in B$ then $a^C = a^{I'}$. (iii): $Gt Alg T = \langle B, H \rangle$ where H is the set of open elements of $Alg T$. Thus $a \in H$ if and only if $a = \max \{b \in G : b \leq a\}$, if and only if $a \in G$ and so $H = G$ as required. (iv): $Alg Gt A$ and A have the same underlying Boolean algebra. Consider $a \in A$. In the proof of (i) above we saw that in A we have $a^I = \max \{b \in A^0 : b \leq a\}$. But $\max \{b \in A^0 : b \leq a\}$ is the interior of a in $Alg Gt A$ and so $Alg Gt A$ has the same interior operator as A and hence also the same closure operator. $\square$

Thus interior algebras and generalized topological spaces are essentially the same things. Moreover, to specify an interior algebra it suffices to specify its underlying Boolean algebra and its set of open elements, that is, its corresponding generalized topological space.

The following is an important proposition relating to the closure of generalised topologies under arbitrary joins:

Proposition 1.5: Let A be an interior algebra and $S \subseteq A^0$. Then the join of S exists in A if and only if the join of S exists in A^0 . If these joins exist then they are equal.

Proof: If b is the join of S in A then $b \in A^0$ since A^0 is a generalized topology in A^u . So b must also be the join of S in A^0 . Conversely, if b is the join of S in A^0 consider an upper bound d of S in A . Then for all $a \in S$, $a \leq d$ and so $a = a^I \leq d^I$. Thus d^I is an upper bound for S in A^0 and so $b \leq d^I \leq d$. It follows that b is also the join of S in A . $\square$

Thus when talking about a join of open elements, we do not have to specify whether the join is taken in A or A^0 . Since an intersection of open subsets of a topological space need not be open we see that the same does not hold for meets of open elements. However, we do have the following easy to prove result:

Proposition 1.6: Let A be an interior algebra and let $S \subseteq A$. Suppose that b is the meet of S in A . Then b^I is the meet of $\{a^I : a \in S\}$ in A^0 . In particular, if $S \subseteq A^0$ then b^I is the meet of S in A^0 . $\square$

Corollary 1.7: If A is a complete interior algebra then A^0 is a complete Heyting algebra. Joins in A^0 coincide with joins in A while meets in A^0 are the interiors of meets in A . $\square$

Definition 1.8: Let B be a Boolean algebra. By a *neighbourhood function* on B we mean a map N from B to the set of filters in B satisfying:

- (i) For all $a \in B$, $\max \{b \in B : a \in N(b)\}$ exists.
 - (ii) For all $a, b \in B$, $a \in N(b)$ if and only if there is a $c \in B$ with $b \leq c \leq a$ and $c \in N(a)$.
- For each $a \in B$, $N(a)$ is called a *neighbourhood filter* at a . If $b \in N(a)$, b is called a *neighbourhood* of a and is said to *surround* a . $\square$

The following proposition follows easily from the definition:

Proposition 1.9: Let N be a neighbourhood function on a Boolean algebra B . Then:

- (i) For all $a \leq b$ in B , $N(b) \subseteq N(a)$.
- (ii) For all $a \in B$, $N(a) \subseteq \{a\}$, and $N(a) = \{a\}$ if and only if $a \in N(a)$. $\square$

Like generalized topological spaces, Boolean algebras equipped with neighbourhood functions provide an alternative description of interior algebras. This is best seen by establishing a duality between generalized topologies and neighbourhood functions.

Definition 1.10: Let B be a Boolean algebra. If N is a neighbourhood function on B put $c(N) = \{b \in B : b \in N(b)\}$ or equivalently $c(N) = \{b \in B : N(b) = \{b\}\}$. If G is a

generalized topology in B define a map $\pi(G)$ on B by $\pi(G)(a) = \{b \in B : a \leq c \leq b \text{ for some } c \in G\}$, for all $a \in B$. $\square$

Theorem 1.11: Let B be a Boolean algebra. Let N be a neighbourhood function on B and let G be a generalized topology in B . Then:

- (i) $\mathcal{G}(N)$ is a generalized topology in B .
- (ii) $\pi(G)$ is a neighbourhood function on B .
- (iii) $\pi\mathcal{G}(N) = N$
- (iv) $\mathcal{G}\pi(G) = G$

Proof: (i): We use Proposition 1.2. Since $N(1)$ is a filter $1 \in N(1)$, and so $1 \in \mathcal{G}(N)$. Let $a, b \in \mathcal{G}(N)$. By Proposition 1.10, $N(a), N(b) \subseteq N(ab)$ and so $a, b \in N(ab)$ whence $ab \in N(ab)$. Thus $ab \in \mathcal{G}(N)$. Hence $\mathcal{G}(N)$ is closed under binary meets. Let $a \in B$. Put $c = \max \{b \in A : a \in N(b)\}$. Then $c \in N(a)$ and so there is a $d \in B$ with $c \leq d \leq a$ and $d \in N(d)$. Then $a \in N(d)$ and so $d \leq c$ whence $c = d$. Thus $c \in N(c)$ whence $c \in \mathcal{G}(N)$. Now consider $e \in \mathcal{G}(N)$ with $e \leq a$. Then $a \in N(e)$ since $e \in N(e)$ and so $e \leq c$. Thus $c = \max \{e \in \mathcal{G}(N) : e \leq a\}$ and the result follows. (ii): Let $a \in B$. $1 \in \pi(G)(a)$ since $1 \in G$ and $a \leq 1 \leq 1$. Suppose $b \in \pi(G)(a)$ and let $c \in B$ with $b \leq c$. Then there is a $d \in G$ with $a \leq d \leq b$ whence $a \leq d \leq c$ and so $c \in \pi(G)(a)$. Now suppose $b, c \in \pi(G)(a)$. There are $d, e \in G$ with $a \leq d \leq b$ and $a \leq e \leq c$. Then $de \in G$ and $a \leq de \leq bc$ whence $bc \in \pi(G)(a)$. Thus $\pi(G)(a)$ is a filter as required. Note that for $d \in B$ we have $d \in \pi(G)(d)$ if and only if there is an $e \in G$ with $d \leq e \leq d$, if and only if $d \in G$. Thus for all $b \in B$ we have $b \in \pi(G)(a)$ if and only if there is a $d \in G$ with $a \leq d \leq b$, if and only if there is a $d \in B$ with $d \in \pi(G)(d)$ and $a \leq d \leq b$. Lastly consider $c = \max \{d \in G : d \leq a\}$. Then $c \in G$ and $c \leq c \leq a$ whence $a \in \pi(G)(c)$. Let $b \in B$ with $a \in \pi(G)(b)$. There is a $d \in G$ with $b \leq d \leq a$. Then $d \leq c$ whence $b \leq c$. Thus $c = \max \{b \in B : a \in \pi(G)(b)\}$. Therefore $\pi(G)$ is a neighbourhood function on B . (iii): Consider $a \in B$. Then for all $b \in B$ we have $b \in \pi\mathcal{G}(N)(a)$ if and only if there is a $d \in \mathcal{G}(N)$ with $a \leq d \leq b$, if and only if there is a $d \in B$ with $d \in N(d)$ and $a \leq d \leq b$, if and only if $b \in N(a)$. Thus $\pi\mathcal{G}(N) = N$. (iv): For all $a \in B$, $a \in \mathcal{G}\pi(G)$ if and only if $a \in \pi(G)(a)$, if and only if there is a $d \in G$ with $a \leq d \leq a$, if and only if $a \in G$. Thus $\mathcal{G}\pi(G) = G$. $\square$

Remark 1.12: From the above theorem we see that pairs $\langle B, N \rangle$, where B is a Boolean algebra and N is a neighbourhood function on B , are essentially the same as things as generalized topological spaces and hence essentially the same things as interior algebras. Given an interior algebra A , the corresponding neighbourhood function $\pi(A^0)$ on A^u is given by $\pi(A^0)(a) = \{b \in A : a \leq b^1\}$, for all $a \in A$. Conversely if N is a neighbourhood function on B , the interior operator of $\text{Alg} \langle B, \mathcal{G}(N) \rangle$ is given by $a^1 = \max \{b \in B : a \in$

$N(b)\}$. $\square$

From now on, when working with an interior algebra A , we will denote the neighbourhood function $\pi(A^0)$ simply by N .

Corollary 1.13: Let A be an interior algebra. For all $a \in A$, $N(a)$ is an open filter. $\square$

However not every open filter is a neighbourhood filter: Let X be a discrete space on an infinite set X and let A be the power set interior algebra determined by X . Let $\mathcal{F}$ be the filter of co-finite subsets of X . Then $\mathcal{F}$ is an open filter in A but it is not the neighbourhood filter of any element of A .

2. Convergence and Accumulation of Subsets of Interior Algebras

Using the concept of neighbourhoods in an interior algebra we can provide an interior algebraic generalization of the topological concepts of system convergence and system accumulation.

Definition 2.1: Let A be an interior algebra. Let $R \subseteq A$ and let $a \in A$. R is said to *converge* to a and a is called a *limit* of R , denoted by $R \rightarrow a$, if and only if for all $b \in N(a)$ there is an $r \in R$ with $r \leq b$. $\square$

Let X be a topological space and let A be the power set interior algebra determined by X . If $x \in X$ and $\mathcal{S} \subseteq \mathcal{P}(X)$ then $\mathcal{S} \rightarrow x$ in X in the usual sense, if and only if $\mathcal{S} \rightarrow \{x\}$ in the interior algebra A . The atoms of A are precisely the singletons $\{x\}$ for $x \in X$ and so we see that Definition 2.1, in the case when a is an atom, is a generalization of system convergence to points. Since we did not restrict the definition to atoms we in fact have a broader generalization. We easily see:

Proposition 2.2: If A is an interior algebra, $R \subseteq A$ and $a \in A$ then the following are all equivalent:

- (i) $R \rightarrow a$
- (ii) For all open $b \geq a$ there is an $r \in R$ with $r \leq b$.
- (iii) For all $b \in N(a)$ there is an $r \in R$ with $r \leq b^1$. $\square$

We remind the reader of some order-theoretic concepts which we will be using. Recall that a *cone* in a poset P , is a subset $S \subseteq P$ such that for all $a, b \in P$, if $a \in S$ and $a \leq b$ then $b \in S$.

A non-empty cone is called a *stack*. If P is a join semi-lattice, a stack S in P is a *grill* if and only if it satisfies the primeness condition: for all $a, b \in P$, $a + b \in S$ implies $a \in S$ or $b \in S$. (In a Boolean algebra, any filter is a stack and any ultrafilter is a grill.) Recall that if $S, R \subseteq P$ we say that S *refines* R if and only if for all $r \in R$ there is an $s \in S$ with $s \leq r$.

If A is an interior algebra and $R \subseteq A$ we put $\lim R = \{a \in A : R \rightarrow a\}$.

Theorem 2.3: Let A be an interior algebra. $\lim$ is a normal weak closure operator on $\mathcal{P}(A)$, that is:

- (a) $R \subseteq \lim R$
- (b) $S \subseteq R$ implies $\lim S \subseteq \lim R$
- (c) $\lim(\lim R) = \lim R$
- (d) $\lim R = \phi$ if and only if $R = \phi$ for all $S, R \subseteq A$.

In addition, for all $S, R \subseteq A$ we have:

- (e) If S refines R then $\lim S \subseteq \lim R$.
- (f) If R is a stack then $\lim R = \{a \in A : N(a) \subseteq R\}$.
- (g) $\lim R$ is a cone. (In particular, $\lim R$ is a stack if and only if $R \neq \phi$.)
- (h) $\lim R = A$ if and only if $0 \in R$.
- (i) $\lim \{1\} = \{a \in A : N(a) = \{1\}\} = \{a \in A : A^0 \cap [a] = \{1\}\}$

Proof: We only prove (c) and leave the rest to the reader. Let $R \subseteq A$. By (a) $\lim R \subseteq \lim(\lim R)$. Let $a \in \lim(\lim R)$. Let $b \in N(a)$. There is a $c \in \lim R$ with $c \leq b^I$. Then $b \in N(c)$ and so there is an $r \in R$ with $r \leq b$. Thus $a \in \lim R$ and so $\lim(\lim R) \subseteq \lim R$. $\square$

Definition 2.4: Let A be an interior algebra. Define a relation $\ll$ on A , by $a \ll b$ if and only if $a \leq b^C$, for all $a, b \in A$. Note that $\ll$ is a pre-order, that is, a reflexive transitive relation. We call $\ll$ the *canonical pre-order* on A . $\square$

If A is an interior algebra we denote the set of atoms of A by $At A$.

Theorem 2.5: Let A be an interior algebra and let $a, b \in At A$. Then:

- (i) $a \ll b$ if and only if $N(a) \subseteq N(b)$.
 - (ii) $a^C = b^C$ if and only if $N(a) = N(b)$.
- Proof:** (i): Let $a \ll b$. Let $d \in N(a)$. Suppose $d \notin N(b)$. Then $b \not\leq d^I$ and so $b \leq d^{I'}$. Thus $a \leq b^C \leq d^{I'C} = d^{I'}$, a contradiction since $a \leq d^I$. Hence $d \in N(b)$ and so $N(a) \subseteq N(b)$. Conversely let $N(a) \subseteq N(b)$. Suppose that $a \not\leq b^C$. Then $a \leq b^{C'} = b'^I$, that is, $b' \in N(a)$. But then $b' \in N(b)$, a contradiction. (ii) follows from (i) since $a^C = b^C$ if and only if $a \ll b$.

and $b \ll a$. $\square$

Definition 2.6: If A is an interior algebra and $a \in A$ we put $E(a) = \{b \in A : a \ll b\}$. The members of $E(a)$ are called *enclosers* of a and are said to *enclose* a . $\square$

Proposition 2.7: Let A be an interior algebra. Then:

- (i) For all $a \in A$, $E(a)$ is a stack.
- (ii) For all $a \in At A$, $E(a)$ is a grill.
- (iii) For all $a \leq b$ in A , $E(b) \subseteq E(a)$.
- (iv) For all $a \in A$, $E(a) = E(a^C)$. $\square$

Note that if $E(a)$ is a grill, a need not be an atom — consider an interior algebra with more than four elements but only 0 and 1 open.

Definition 2.8: Let A be an interior algebra. The enclosers of 1 in A are called the *dense* elements of A . (In other words, $a \in A$ is *dense* if and only if $a^C = 1$.)

Recall that an element of a Heyting algebra is called *dense* if and only if its pseudocomplement is the bottom element 0. We have the following neat result:

Theorem 2.9: Let A be an interior algebra. Then the open dense elements of A are precisely the dense elements of the Heyting algebra A^0 .

Proof: Let $a \in A^0$. Then a is a dense element of A , if and only if $a^C = 1$, if and only if $a^{C'} = 0$, if and only if $a'^I = 0$. Noting that a'^I is the pseudocomplement of a in A^0 , the result follows. $\square$

Definition 2.10: Let A be an interior algebra, $R \subseteq A$ and $a \in A$. We say that R *accumulates* at a and that a is an *accumulant* of R , denoted by $R \dashv a$, if and only if $R \subseteq E(a)$. $\square$

Note that in the case when a is an atom the definition of accumulation is a generalization of the topological concept of system accumulation at points. If A is an interior algebra and $R \subseteq A$ then we put $\text{acc } R = \{a \in A : R \dashv a\}$.

Proposition 2.11: Let A be an interior algebra and let $S, R \subseteq A$. Then:

- (i) $\text{acc } R$ is a closed dual stack.
- (ii) $S \subseteq R$ implies $\text{acc } R \subseteq \text{acc } S$.

- (iii) $\text{acc } R = \{0\}$ if and only if $0 \in R$.
 (iv) $\text{acc } R = A$ if and only if $R \subseteq \{1\}$. $\square$

Definition 2.12: Let L be a meet semi-lattice with 0. If $R \subseteq A$ we define the *section* of R , denoted by $\text{Sec } R$, to be the set $\{b \in L : ab > 0 \text{ for all } a \in R\}$. This gives a mapping $\text{Sec} : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ known as the *section operator*. $\square$

The section operator has been used before in the literature in the special case where L is the power set of a set. The following proposition summarizes the basic properties of Sec .

Proposition 2.13: Let L be a meet semi-lattice with 0 and let $S, R \subseteq L$. Then:

- (i) $S \subseteq R$ implies $\text{Sec } R \subseteq \text{Sec } S$.
 (ii) $R \subseteq \text{Sec } S$ if and only if $S \subseteq \text{Sec } R$.
 (iii) $R \subseteq \text{Sec } (\text{Sec } R)$.
 (iv) $\text{Sec } R = \emptyset$ if and only if $0 \in R$.
 (v) $\text{Sec } R$ is a cone. (In particular, $\text{Sec } R$ is a stack if and only if $0 \notin R$.) $\square$

We can characterize dense elements using the section operator:

Theorem 2.14: Let A be an interior algebra. Then $\text{Sec } (A^0 - \{0\})$ is the set of all dense elements of A .

Proof: Let a be dense in A . Let $b \in A^0 - \{0\}$. Suppose $ab = 0$. Then $a \leq b'$ whence $1 = b'^c = b'$ and so $b = 0$, a contradiction. Thus $ab > 0$ and so $a \in \text{Sec } (A^0 - \{0\})$. Conversely let $a \in \text{Sec } (A^0 - \{0\})$. Suppose $a^c < 1$. Then $a^{c'} \in A^0 - \{0\}$ and so $aa^{c'} > 0$, a contradiction. Thus $a^c = 1$, that is, a is dense. $\square$

The section operator allows us to establish a connection between neighbourhoods and enclosers and between convergence and accumulation.

Proposition 2.15: Let A be an interior algebra and let $a > 0$ in A . Then $E(a) \subseteq \text{Sec } N(a)$ and $N(a) \subseteq \text{Sec } E(a)$.

Proof: Let $b \in N(a)$ and $d \in E(a)$. Suppose $bd = 0$. Then $b \leq d'$ and so $b^I \leq d'^I$. But $a \leq b^I$ and so $d^c = d'^I \leq a'$. Then $a \leq d^c \leq a'$, a contradiction. Thus $bd > 0$ and the result follows. $\square$

Theorem 2.16: Let A be an interior algebra, let $R \subseteq A$ and let $a \in \text{At } A$. Then:

- (i) $E(a) = \text{Sec } N(a)$ and $N(a) = \text{Sec } E(a)$.
 (ii) $R \rightarrow a$ implies $\text{Sec } R \leftarrow a$.
 (iii) $\text{Sec } R \rightarrow a$ implies $R \leftarrow a$.

Proof: (i): By Proposition 2.15 it suffices to show $\text{Sec } N(a) \subseteq E(a)$. Suppose there is a $b \in \text{Sec } N(a)$ with $b \notin E(a)$. Then $a \not\leq b^c$ and so $a \leq b^{c'} = b'^I$. Hence $b' \in N(a)$. However $bb' = 0$, contradicting $b \in \text{Sec } N(a)$. (ii): Let $R \rightarrow a$. Suppose we do not have $\text{Sec } R \leftarrow a$. Then there is a $b \in \text{Sec } R$ with $a \not\leq b^c$. Then $a \leq b^{c'} = b'^I$. Hence there is an $r \in R$ with $r \leq b'$. Then $rb = 0$, contradicting $b \in \text{Sec } R$. (iii): Let $\text{Sec } R \rightarrow a$. By (ii) we have $\text{Sec } (\text{Sec } R) \leftarrow a$. But $R \subseteq \text{Sec } (\text{Sec } R)$ and so $R \leftarrow a$. $\square$

Note that (i) above does not characterize atoms — consider an interior algebra with more than four elements but only 0 and 1 open.

Proposition 2.17: Let A be an interior algebra, let $a > 0$ in A and let G be a grill in A . Then $G \leftarrow a$ implies $G \rightarrow a$.

Proof: Suppose $G \leftarrow a$. Let $b \in N(a)$. Suppose $b \notin G$. Since G is a grill, $b' \in G$. Then $a \leq b'^c = b^I$. But $a \leq b^I$ and so $a = 0$, a contradiction. Thus $b \in G$. Therefore $N(a) \subseteq G$ and the result follows. $\square$

Corollary 2.18: Let A be an interior algebra. Let $F \subseteq A$ be a proper filter and let $a \in \text{At } A$. Then:

- (i) $F \rightarrow a$ implies $F \leftarrow a$.
 (ii) If F is an ultrafilter, $F \rightarrow a$ if and only if $F \leftarrow a$.

Proof: (i): Let $F \rightarrow a$. By Theorem 2.16 (ii) $\text{Sec } F \leftarrow a$. Since F is a proper filter $F \subseteq \text{Sec } F$ and so $F \leftarrow a$. (ii) follows from (i) and Proposition 2.17. $\square$

Proposition 2.19: Let A be an interior algebra and let $a, b \in A$ with $a > 0$. Consider the following statements:

- (i) $a \ll b$
 (ii) There is a proper (ultra)filter F of $(b]$ with $F \rightarrow a$ in A .
 (iii) There is an $F \subseteq (b]$ with $0 \notin F$ and $F \rightarrow a$ in A .

We always have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If a is an atom then (i), (ii) and (iii) are all equivalent.

Proof: Assume (i). Put $R = \{bd : d \in N(a)\}$. Then $R \subseteq (b]$. By Proposition 2.15, $b \in \text{Sec } N(a)$. Thus $0 \notin R$. Since $N(a)$ is closed under binary meets, so is R . Thus there is an ultrafilter F of $(b]$ with $R \subseteq F$. If $d \in N(a)$ then $bd \in F$ and $bd \leq d$. Thus $F \rightarrow a$ and so (i) $\Rightarrow$ (ii). Clearly (ii) $\Rightarrow$ (iii). Now let a be an atom. Assume (iii). Let $d \in N(a)$. Then there is

a $c \in F$ with $c \leq d$. Then $0 < c = bc \leq bd$. Hence $b \in \text{Sec } N(a)$. By Theorem 2.16 (i) $a \ll b$. Thus (iii) $\Rightarrow$ (i). $\square$

3. Nets and Filter Bases in Interior Algebras

Recall that a *directed set* $W = \langle W, \sqsubseteq \rangle$, is a pair with W a set and $\sqsubseteq$ a pre-order on W with the property that for all $x, y \in W$ there is a $z \in W$ with $x \sqsubseteq z$ and $y \sqsubseteq z$. If W is non-empty and X is a set then a *net in X based on W* is a family (x_i) of elements of X indexed by W . Note that any sequence (x_n) in X can be regarded as a net based on the directed set $\langle \omega, \leq \rangle$.

Definition 3.1: Let (x_i) be a net in a set X based on a directed set W . If $j \in W$ then the set $\{x_i : i \sqsupseteq j \text{ in } W\}$ is called a *tail* of (x_i) and is denoted by $t_j(x_i)$. $\square$

Suppose that A is an interior algebra and $B \subseteq A$. Then $\langle B, \geq \rangle$ is a directed set if and only if B is a filter base in A . If B is a filter base we thus have a net (a_i) in A based on $\langle B, \geq \rangle$ with $a_i = i$ for all $i \in B$.

Definition 3.2: If B is a filter base in an interior algebra A , the net (a_i) given by $a_i = i$ for all $i \in B$, will be called the *canonical net* of B . $\square$

Remark 3.3: Let A be an interior algebra. If (a_i) is a net in A based on a directed set W and if $\Sigma t_j(a_i)$ exists for all $j \in W$ then $\{\Sigma t_j(a_i) : j \in W\}$ is a filter base. Every filter base in A is obtained in this way from its canonical net. $\square$

Recall that a filter base B is called *proper* if and only if $0 \notin B$.

Theorem 3.4: Let B be a proper filter base in an atomic interior algebra A . Then there is a net (a_i) in $\text{At } A$ based on a directed set W such that $B = \{\Sigma t_j(a_i) : j \in W\}$.

Proof: Put $W = \{(c, b) : b \in B \text{ and } c \in \text{At } A \text{ with } c \leq b\}$. There is a $b \in B$. Since B is proper $b > 0$. Since A is atomic there is an atom $c \leq b$. Then $(c, b) \in W$ and so $W \neq \emptyset$. Define $\sqsubseteq$ on W by $(c, b) \sqsubseteq (d, a)$ if and only if $b \geq a$. Then clearly $\sqsubseteq$ is a pre-order. If $(c, b), (d, a) \in W$ then $a, b \in B$ and so there is an $e \in B$ with $e \leq ab$. Since B is proper, $e > 0$ and so by atomicity there is an atom $v \leq e$. Then $(v, e) \in W$, $(c, b) \sqsubseteq (v, e)$ and $(d, a) \sqsubseteq (v, e)$. Thus $W = \langle W, \sqsubseteq \rangle$ is a directed set. If $i = (c, b) \in W$ put $a_i = c$. We thus have a net (a_i) in $\text{At } A$. Consider a $j \in W$ with $j = (d, e)$. Let $i \sqsupseteq j$ in W with $i = (d, e)$. Then $a_i \leq e \leq b$. Thus $t_j(a_i) \subseteq \text{At } (b)$. Now suppose d is an atom with $d \leq b$. Then $(d, b) \sqsupseteq j$ and so $d \in t_j(a_i)$.

Thus in fact $t_j(a_i) = \text{At } (b)$. Hence $b = \Sigma t_j(a_i)$ and so we see that $\{\Sigma t_j(a_i) : j \in W\} \subseteq B$. Lastly consider a $b \in B$. As before there is an atom $c \leq b$. Put $j = (c, b)$. Then $j \in W$ and as above we see that $b = \Sigma t_j(a_i)$. Thus in fact $B = \{\Sigma t_j(a_i) : j \in W\}$. $\square$

What sort of filter bases correspond to sequences?

Definition 3.5: Let A be an interior algebra and let B be a filter base in A . B is called a *sequential base* if and only if $\langle B, \leq \rangle \cong \langle \tau, \geq \rangle$ for some $r \leq \omega$. $\square$

For example, if A is an interior algebra and (a_n) is a sequence in A such that the join $\Sigma t_k(a_n)$ exists for all $k < \omega$, then $\{\Sigma t_k(a_n) : k < \omega\}$ is a sequential base.

Definition 3.6: A sequential base B in an interior algebra A is called *elementary* if and only if the top element of B is a countable non-empty join of atoms and for all $a, b \in B$, a covers b in $\langle B, \leq \rangle$ if and only if a covers b in A . $\square$

The following theorem shows that the elementary sequential bases are precisely the filter bases obtained from sequences of atoms. Although the result is fairly intuitive it is quite tricky to prove correctly and so we have given a fairly detailed proof.

Theorem 3.7: Let A be an interior algebra and let $B \subseteq A$. Then B is an elementary sequential base if and only if there is a sequence (a_n) in $\text{At } A$ with $B = \{\Sigma t_k(a_n) : k < \omega\}$.

Proof: Let B be an elementary sequential base. There is an $r \leq \omega$ such that $\langle B, \leq \rangle \cong \langle \tau, \geq \rangle$. (Note that $r \geq 1$ since a filter base cannot be empty.) Let $\{b_n : n < r\}$ be the enumeration of B given by the isomorphism. There are two cases: Case (1): $r < \omega$. We define atoms c_n , $n < r - 1$, by finite induction such that for all $n < r - 1$, $b_{n+1} = b_0 \cdot c_0' \cdots c_n'$. b_0 is a join of atoms so in fact $b_0 = \Sigma \text{At } (b_0)$. If $r = 1$ let c_0 be any atom below b_0 . Otherwise $b_1 < b_0$ and so there is an atom $c_0 \in (b_0) - (b_1)$. Then $b_1 \leq c_0'$ and so $b_1 \leq b_0 \cdot c_0'$. Since b_0 covers b_1 we in fact have $b_1 = b_0 \cdot c_0'$. Now suppose for some $k < r - 2$ we have found $c_0, \dots, c_k$ with the desired property. We have $b_{k+1} = b_0 \cdot c_0' \cdots c_k' = \Sigma (\text{At } (b_0) - \{c_0, \dots, c_k\})$. $b_{k+2} < b_{k+1}$ and so there is a $c_{k+1} \in \text{At } (b_0) - \{c_0, \dots, c_k\}$ not below b_{k+2} . Then $b_{k+2} \leq c_{k+1}'$ and so $b_{k+2} \leq b_{k+1} \cdot c_{k+1}'$. But b_{k+1} covers b_{k+2} and so in fact $b_{k+2} = b_{k+1} \cdot c_{k+1}' = b_0 \cdot c_0' \cdots c_{k+1}'$ and the induction is complete. Put $c_n = c_{r-2}$ for all $n \geq r - 1$ to obtain (c_n) . There are two cases now: (i): $\text{At } (b_0) = \{c_n : n < r - 1\}$. Then simply put $(a_n) = (c_n)$. Then $b_0 = \Sigma \text{At } (b_0) = \Sigma t_0(a_n)$. If $k < r - 1$ then $b_{k+1} = b_0 \cdot c_0' \cdots c_k' = \Sigma (\text{At } (b_0) - \{c_0, \dots, c_k\}) = \Sigma t_{k+1}(a_n)$. If $k \geq r - 1$ then $\Sigma t_k(a_n) = \Sigma t_{r-2}(a_n) = b_{r-2}$. Thus $B = \{\Sigma t_k(a_n) :$

$k < \omega$. (ii): $\{c_n : n < r-1\} \in \text{At}(b_0)$. Then $\text{At}(b_0) - \{c_n : n < r-1\}$ is countable and non-empty. Let $\{d_n : n < \omega\} = \text{At}(b_0) - \{c_n : n < r-1\}$. Define $\{a_n\}$ as follows: Consider the polynomial $f(n) = (n^2 + 3n)/2$. Let $n < \omega$. If $k = f(n)$ for some $n < \omega$ put $a_k = c_n$. If $f(n) < k < f(n+1)$ for some $n < \omega$ put $a_k = d_i$ where $i = k - f(n) - 1$. Observe that $f(n+1) = f(n) + n + 2$ for all $n < \omega$. Let $k < \omega$. There is an $j < \omega$ such that $k \leq f(j)$. Consider $s < \omega$. Put $n = j + s$. Then $0 \leq s < n+1 = f(n+1) - f(n) - 1$. Hence $f(n) < f(n) + 1 \leq s + f(n) + 1 < f(n+1)$ and so $a_s = d_i$ where $i = s + f(n) + 1$. Also $k < f(j) < f(n) < i$. Thus $d_i \in t_k(a_n)$. Hence for all $k < \omega$, $\{d_n : n < \omega\} \subseteq t_k(a_n)$. Then $\text{At}(b_0) = \{c_n : n < r-1\} \cup \{d_n : n < \omega\} = \{c_n : n < \omega\} \cup \{d_n : n < \omega\} = t_0(a_n)$ and so $b_0 = \sum t_0(a_n)$. Let $k < r-1$. Put $i = f(k+1)$. Then $b_{k+1} = b_0 \cdot c_0' \cdots c_k' = \sum (\text{At}(b_0) - \{c_0, \dots, c_k\}) = \sum t_i(a_n)$. Let $k \geq r-1$. Put $i = f(k+1)$ and $j = f(i-2)$. Then $\sum t_i(a_n) = \sum t_j(a_n)$. Lastly let $k < \omega$ and suppose $f(k) < i < f(k+1)$. Then $\sum t_i(a_n) = \sum t_j(a_n)$ where $j = f(k)$. It follows that $B = \{\sum t_k(a_n) : k < \omega\}$. Case (2): $u = \omega$. Then we proceed as in (1) but obtain the whole of $\{c_n\}$ by induction. We proceed as in (1) but consider $k < \omega$ instead of $k < r-1$ and no longer consider $k \geq r-1$. This completes the forward implication. For the converse: Let $\{a_n\}$ be a sequence such that $B = \{\sum t_k(a_n) : k < \omega\}$. Then B is a sequential base with top element $\sum t_0(a_n)$, a countable non-empty join of atoms. Obviously if $a, b \in B$ and a covers b in A then a covers b in $\langle B, \leq \rangle$. Suppose that a covers b in $\langle B, \leq \rangle$. There are $i < j < \omega$ with $a = \sum t_j(a_n)$ and $b = \sum t_i(a_n)$. Let $k \leq j$ be least such that $a = \sum t_k(a_n)$. Then $b \leq \sum t_{k-1}(a_n) < a$ and so $b = \sum t_{k-1}(a_n)$. Thus $a = b + a_k$ and a_k is an atom whence it follows that a covers b in A . Thus B is elementary. $\square$

Definition 3.8: Call a filter in an interior algebra, *sequential*, if and only if it has a countable base. $\square$

The following proposition justifies the above definition.

Proposition 3.9: A filter in an interior algebra has a sequential base if and only if it is sequential.

Proof: Let F be a filter in an interior algebra A . If $\{b_n : n < \omega\}$ is a countable base for F , define $d_n = b_0 \cdots b_n$ for all $n < \omega$. Then $\{d_n : n < \omega\}$ is a sequential base for F . On the other hand, any sequential base is itself a countable base. $\square$

Definition 3.10: Call a filter in an interior algebra, *elementary*, if and only if it has an elementary sequential base. $\square$

Note that the intersection of two sequential filters is sequential since if F and G are filters with bases B and C respectively then $\{b + c : b \in B \text{ and } c \in C\}$ is a base for $F \cap G$. We now show that a similar result holds for elementary filters.

Theorem 3.11: Let F and G be elementary filters in an interior algebra A . Then $F \cap G$ is elementary.

Proof: There are sequences $\{a_n\}$ and $\{b_n\}$ in $\text{At } A$ such that $B = \{\sum t_k(a_n) : k < \omega\}$ and $C = \{\sum t_k(b_n) : k < \omega\}$ are bases for F and G respectively. Define $\{c_n\}$ by $c_{2n} = a_n$ and $c_{2n+1} = b_n$ for all $n < \omega$. For all $k < \omega$, $t_{2k}(c_n) = t_k(a_n) \cup t_k(b_n)$ and so $\sum t_{2k}(c_n) = \sum t_k(a_n) + \sum t_k(b_n)$. If $d \in F \cap G$ there are $i, j < \omega$ such that $\sum t_i(a_n) \leq d$ and $\sum t_j(b_n) \leq d$. Let $k = i + j$. Then $\sum t_{2k}(c_n) = \sum t_k(a_n) + \sum t_k(b_n) \leq \sum t_i(a_n) + \sum t_j(b_n) \leq d$. Conversely, if $\sum t_{2k}(c_n) \leq d$ for some $k < \omega$, then $\sum t_k(a_n) \leq d$ and $\sum t_k(b_n) \leq d$ whence $d \in F \cap G$. Thus $\{\sum t_{2k}(c_n) : k < \omega\}$ is a base for $F \cap G$. Also, for all $k < \omega$, $t_{2k+1}(c_n) = t_{2k+2}(c_n) \cup \{b_k\}$ whence $\sum t_{2k+1}(c_n) = \sum t_{2k+2}(c_n) + b_k \geq \sum t_{2k+2}(c_n)$. It follows that $\{\sum t_k(c_n) : k < \omega\}$ is a base for $F \cap G$. $\square$

We can convert nets into filters: Let $\{a_i\}$ be a net in an interior algebra A based on a directed set W . Put $F(a_i) = \{b \in A : t_j(a_i) \subseteq \langle b \rangle \text{ for some } j \in W\}$. Clearly $F(a_i)$ is a stack. Let $b, c \in F(a_i)$. There are $j, k \in W$ with $t_j(a_i) \subseteq \langle b \rangle$ and $t_k(a_i) \subseteq \langle c \rangle$. There is an $r \in W$ with $j \subseteq r$ and $k \subseteq r$. Then $t_r(a_i) \subseteq \langle b \rangle \cap \langle c \rangle = \langle bc \rangle$. Thus $bc \in F(a_i)$ and so $F(a_i)$ is a filter. Moreover, every filter in A is of this form since if F is a filter in A , $F = F(a_i)$ where $\{a_i\}$ is the canonical net of F . As a corollary to Theorem 3.4 we have the following result:

Corollary 3.12: Let F be a proper filter in an atomic interior algebra A . Then there is a net $\{a_i\}$ in $\text{At } A$ based on a directed set W , with $F = F(a_i) = \{\sum t_j(a_i) : j \in W\}$. $\square$

Definition 3.13: Call a filter F in an interior algebra A , *pseudo-elementary* if and only if there is a sequence $\{a_n\}$ in $\text{At } A$ such that $F = F(a_n)$. $\square$

Definition 3.14: Call an interior algebra A , *countably collectable* if and only if $\sum R$ exists for every countable $R \subseteq \text{At } A$. $\square$

Recall that a net $\{x_i\}$ is called *injective* if and only if the map $i \mapsto x_i$ is injective.

Theorem 3.15: Let A be an interior algebra and let F be an ultrafilter in A . Consider the following statements:

(i) F is principal.

(ii) F is elementary.

(iii) F is sequential.

(iv) F is pseudo-elementary.

Then we always have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (iv). If A is countably collectable then (i), (ii) and (iv) are all equivalent. If A is atomic then (iii) $\Rightarrow$ (iv). In particular, if A is atomic and countably collectable then (i) – (iv) are all equivalent.

Proof: Recall that if F is a principal ultrafilter then $F = [a]$ for some atom a . Putting $a_n = a$ for all $n < \omega$ gives $\{\Sigma t_k(a_n) : n < \omega\} = \{a\}$ and so F is elementary. Thus (i) $\Rightarrow$ (ii). Obviously (ii) $\Rightarrow$ (iii). If F is elementary then there is a sequence (a_i) in $At A$ such that $\{\Sigma t_k(a_i) : k \leq \omega\}$ is a base for F . But then $F = F(a_i)$. Thus (ii) $\Rightarrow$ (iv). Let A be countably collectable. Suppose that there is a sequence (a_n) in $At A$ with $F = F(a_n)$. Then for all $k < \omega$, $\Sigma t_k(a_n)$ exists and $\{\Sigma t_k(a_n) : k < \omega\}$ is a base for F . Thus (iv) $\Rightarrow$ (ii). Assume (ii). There is a sequence (a_n) in $At A$ such that $B = \{\Sigma t_k(a_n) : k < \omega\}$ is a base for F . Suppose that $\{a_n : n < \omega\}$ is infinite. Define a subsequence (b_k) of (a_n) inductively as follows: Put $m(0) = 0$. Suppose that for some $k < \omega$ we have defined $m(0) < \dots < m(k)$ so that $a_{m(0)}, \dots, a_{m(k)}$ are all distinct. Since $\{a_n : n < \omega\}$ is infinite there is an $m(k+1) > m(k)$ such that $a_{m(0)}, \dots, a_{m(k+1)}$ are all distinct. We thus have a strictly increasing map $m : \omega \rightarrow \omega$ and hence a subsequence $(b_k) = (a_{m(k)})$ of (a_n) . In fact (b_k) is injective. Since A is countably collectable we can put $c = \Sigma \{b_{2k} : k < \omega\}$. Suppose $c \in F$. Then there is an $i < \omega$ with $\Sigma t_i(a_n) \leq c$. Then there is a $k < \omega$ with $b_{2k+1} \in t_i(a_n)$. Thus $b_{2k+1} \leq c$ and so $b_{2k+1} = b_{2n}$ for some $n < \omega$. But then $2k+1 = 2n$, a contradiction. Thus $c \notin F$. But F is an ultrafilter and so $c' \in F$. Thus there is an $i < \omega$ with $\Sigma t_i(a_n) \leq c'$. Then there is a $k < \omega$ with $b_{2k} \in t_i(a_n)$. Then $b_{2k} \leq c'$, a contradiction since $b_{2k} \leq c$. Hence the assumption that $\{a_n : n < \omega\}$ is infinite is false. Thus B is finite and so F is principal. Thus (ii) $\Rightarrow$ (i). Now let A be atomic. Assume (iii). Then there is a base B for F with $\langle B, \leq \rangle \cong \langle r, \geq \rangle$ for some $r \leq \omega$. If $r < \omega$ then (i) and hence (iv) holds. Suppose $r = \omega$. Let $\{b_n : n < \omega\}$ be the enumeration of B given by the isomorphism between $\langle B, \leq \rangle$ and $\langle \omega, \geq \rangle$. Then for all $n < \omega$, $b_n > 0$ and so there is an atom $a_n \leq b_n$. Let $k < \omega$. If $n \geq k$ then $a_n \leq b_n \leq b_k$ and so $t_k(a_n) \leq (b_k)$. Thus $F \subseteq F(a_n)$. Since F is an ultrafilter and $F(a_n)$ is proper we have $F = F(a_n)$. Thus (iii) $\Rightarrow$ (iv). $\square$

In the above theorem we saw that a sequential ultrafilter in an atomic interior algebra is pseudo-elementary. In the case of open ultrafilters we can make do with a property strictly weaker than atomicity.

Definition 3.16: An interior algebra A is called open atomic if and only if for all open $a > 0$

there is an atom b of A with $b \leq a$. $\square$

(To see that open atomicity is weaker than atomicity consider an interior algebra A , with underlying Boolean algebra isomorphic to the product of the free Boolean algebra on $\aleph_0$ generators and the two element Boolean algebra, and with $A^0 = \{0,1\}$. Then A is open atomic but not atomic.)

Theorem 3.17: Let A be an open atomic interior algebra. If F is an open sequential ultrafilter in A then F is pseudo-elementary.

Proof: Let F be an open ultrafilter in A . Then there is a sequential base B for F . There is an $r \leq \omega$ with $\langle B, \leq \rangle \cong \langle r, \geq \rangle$. If $r < \omega$ then F is principal and hence pseudo-elementary. Suppose $r = \omega$. Let $\{b_n : n < \omega\}$ be the enumeration of B given by the isomorphism between $\langle B, \leq \rangle$ and $\langle \omega, \geq \rangle$. For all $n < \omega$, $b_n \in F$ and so $b_n^I > 0$ whence by open atomicity there is an atom $a_n \leq b_n^I$. Let $k < \omega$. For all $n > k$, $a_n \leq b_n^I \leq b_k^I \leq b_k$ and so $t_k(a_n) \leq (b_k)$. Thus $F \subseteq F(a_n)$. Since F is an ultrafilter and $F(a_n)$ is proper, $F = F(a_n)$ and so F is pseudo-elementary. $\square$

Corollary 3.18: Let A be an open atomic countably collectable interior algebra. Then the following are equivalent for an open ultrafilter F in A :

- (i) F is principal.
- (ii) F is elementary.
- (iii) F is sequential.
- (iv) F is pseudo-elementary. $\square$

4. Convergence and Accumulation of Nets and Sequences

Definition 4.1: Let A be an interior algebra, let (a_i) be a net in A based on a directed set W and let $b \in A$. We say that (a_i) converges to b and that b is a limit of (a_i) , denoted by $(a_i) \rightarrow b$, if and only if for all $d \in N(b)$ there is a $j \in W$ such that $a_i \leq d$ for all $i \geq j$ in W . $\square$

Of course in the case of atoms, net convergence in interior algebras is a generalization of net convergence in topology.

Proposition 4.2: Let A be an interior algebra, let (a_i) be a net in A based on a directed set W and let $b \in A$. Then the following are equivalent:

- (i) $(a_i) \rightarrow b$
- (ii) For all open $d \geq b$ there is a $j \in W$ such that $a_i \leq d$ for all $i \geq j$ in W .

(iii) For all $d \in N(b)$ there is a $j \in W$ such that $a_i \leq d^I$ for all $i \supseteq j$ in W . $\square$

Put $\lim (a_i) = \{b \in A : (a_i) \rightarrow b\}$.

Proposition 4.3: Let A be an interior algebra and let (a_i) be a net in A . Then:

(i) $\lim (a_i)$ is a stack.

(ii) If (b_k) is a net in $\lim (a_i)$ then $\lim (b_k) \subseteq \lim (a_i)$.

Proof: (i) is trivial, we prove (ii): Let (b_k) be a net in $\lim (a_i)$ based on a directed set Z and let W be the directed set on which (a_i) is based. Let $c \in A$ with $(b_k) \rightarrow c$. We have to show that $(a_i) \rightarrow c$. Let $d \in N(c)$. Then there is an $r \in Z$ with $b_k \leq d^I$ for all $k \supseteq r$ in Z , in particular $b_r \leq d^I$. Then $d \in N(b_r)$. But then there is a $j \in W$ with $a_i \leq d$ for all $i \supseteq j$ in W . $\square$

How is net convergence related to subset convergence?

Proposition 4.4: If (a_i) is a net in an interior algebra A and $b \in A$ then $(a_i) \rightarrow b$ if and only if $F(a_i) \rightarrow b$. In other words $\lim (a_i) = \lim F(a_i)$. $\square$

Proof: Let W be the net on which (a_i) is based. Consider $d \in N(b)$. Then there is a $j \in W$ such that $a_i \leq d$ for all $i \supseteq j$ in W , if and only if there is a $j \in W$ with $t_j(a_i) \subseteq \{d\}$, if and only if $d \in F(a_i)$. Thus $(a_i) \rightarrow b$ if and only if $N(b) \subseteq F(a_i)$ and the result follows. $\square$

Proposition 4.5: Let A be an interior algebra and let (a_i) be a net in A based on a directed set W . Suppose that for all $j \in W$ the join $\Sigma t_j(a_i)$ exists. Then $\lim (a_i) = \lim \{\Sigma t_j(a_i) : j \in W\}$.

Proof: For all $b \in A$ we have $(a_i) \rightarrow b$ if and only if for all $d \in N(b)$ there is a $j \in W$ with $a_i \leq d$ for all $i \supseteq j$ in W , if and only if for all $d \in W$ there is a $j \in W$ with $\Sigma t_j(a_i) \leq d$, and the result follows. $\square$

Remark 4.6: Proposition 4.4 tells us that net convergence can be considered to be a special case of filter convergence. Proposition 4.5 together with Remark 3.3 show that filter base convergence can be considered to be a special case of net convergence. But filter convergence is a special case of filter base convergence. Thus filter convergence, filter base convergence and net convergence are all equivalent in a sense. $\square$

Theorem 4.7: Let A be an interior algebra, let $R \subseteq A$ and let $b \in A$. Then $R \rightarrow b$ if and only if there is a net (a_i) in R with $(a_i) \rightarrow b$.

Proof: Suppose $R \rightarrow b$. We define a net in R based on the directed set $\langle N(b), \supseteq \rangle$ as

follows: For each $i \in N(b)$ let $a_i \in R$ with $a_i \leq i$. Then if $d \in N(b)$, for all $i \leq d$ in $\langle N(b), \supseteq \rangle$ we have $a_i \leq i \leq d$. Thus $(a_i) \rightarrow b$. Conversely, suppose there is a net (a_i) in R based on a directed set W with $(a_i) \rightarrow b$. Let $d \in N(b)$. Then there is a $j \in W$ with $a_i \leq d$ for all $i \supseteq j$ in W , in particular, $a_j \leq d$. Thus $R \rightarrow b$. $\square$

Definition 4.8: Let A be an interior algebra, let (a_i) be a net in A based on a directed set W and let $b \in A$. We say that (a_i) *accumulates* at b and that b is an *accumulant* of (a_i) , denoted by $(a_i) \leftarrow b$, if and only if for all $d \in N(b)$ and for all $j \in W$, there is an $i \supseteq j$ in W with $a_i \leq d$. $\square$

Of course in the case of atoms, net accumulation in interior algebras generalizes net accumulation in topology.

Proposition 4.9: Let A be an interior algebra, let (a_i) be a net in A based on a directed set W and let $b \in A$. Then the following are equivalent:

(i) $(a_i) \leftarrow b$

(ii) For all open $d \geq b$ and for all $j \in W$, there is an $i \supseteq j$ with $a_i \leq d$.

(iii) For all $d \in N(b)$ and for all $j \in W$, there is an $i \supseteq j$ with $a_i \leq d^I$. $\square$

Put $\text{acc}(a_i) = \{b \in A : (a_i) \leftarrow b\}$.

Proposition 4.10: Let A be an interior algebra and let (a_i) be a net in A .

(i) $\text{acc}(a_i)$ is a stack.

(ii) $\lim (a_i) \subseteq \text{acc}(a_i)$.

(iii) If (b_k) is a net in $\text{acc}(a_i)$ then $\text{acc}(b_k) \subseteq \text{acc}(a_i)$.

Proof: (i) and (ii) are trivial. For (iii): Let (b_k) be a net in $\text{acc}(a_i)$ based on a directed set Z and let W be the directed set on which (a_i) is based. Let $c \in A$ with $(b_k) \leftarrow c$. We have to show that $(a_i) \leftarrow c$. Let $d \in N(c)$ and let $j \in W$. There is an $r \in Z$. (By the definition of a net Z must be non-empty.) Then there is a $k \supseteq r$ with $b_k \leq d^I$. Then $d \in N(b_k)$. Hence there is an $i \supseteq j$ with $a_i \leq d$. $\square$

Definition 4.11: Let A be an interior algebra and let (a_i) be a net in A based on a directed set W . Then we say that (a_i) is *non-zero* if and only if $a_i > 0$ for all $i \in W$. $\square$

Net accumulation is generally very different to subset accumulation. However we have the following results:

Proposition 4.12: Let A be an interior algebra, let (a_i) be a non-zero net in A based on a directed set W . Then:

- (i) $\text{At } A \cap \text{acc } (a_i) \subseteq \text{acc } F(a_i)$.
- (ii) Suppose that for all $j \in W$ the join $\Sigma t_j(a_i)$ exists. Then $\text{At } A \cap \text{acc } (a_i) \subseteq \text{acc } \{\Sigma t_j(a_i) : j \in W\}$.

Proof: (i): Suppose b is an atom in A with $(a_i) \rightarrow b$. Consider $c \in F(a_i)$. Let $d \in N(b)$. There is a $j \in W$ with $t_j(a_i) \subseteq [c]$. There is an $i \supseteq j$ in W with $a_i \leq d$. Then $a_i \leq cd$ and so $cd > 0$ since (a_i) is non-zero. Hence $c \in \text{Sec } N(b) = E(b)$ (See Theorem 2.16 (i).) Thus $F(a_i) \rightarrow b$ as required. (ii) follows from (i) since if $\Sigma t_j(a_i)$ exists for all $j \in W$ then $\{\Sigma t_j(a_i) : j \in W\} \subseteq F(a_i)$. $\square$

Proposition 4.13: Let A be an atomic interior algebra and let (a_i) be a net in A based on a directed set W . Suppose that for all $j \in W$ the join $\Sigma t_j(a_i)$ exists. Then $\text{At } A \cap \text{acc } (a_i) = \text{At } A \cap \text{acc } \{\Sigma t_j(a_i) : j \in W\}$.

Proof: Let $b \in \text{At } A$ with $\{\Sigma t_j(a_i) : j \in W\} \rightarrow b$. Let $d \in N(b)$ and let $j \in W$. Then $\Sigma t_j(a_i) \in E(b)$. But $E(b) = \text{Sec } N(b)$ (by Theorem 2.16 (i)) and so $d \cdot \Sigma t_j(a_i) > 0$ whence there is an atom $c \leq d \cdot \Sigma t_j(a_i)$. But then $c = a_i$ for some $i \supseteq j$ in W and then $a_i \leq d$. Thus $(a_i) \rightarrow b$. Hence $\text{At } A \cap \{\Sigma t_j(a_i) : j \in W\} \subseteq \text{At } A \cap \text{acc } (a_i)$. The reverse inclusion follows from Proposition 4.12 (ii). $\square$

Note that Proposition 4.10 (ii) already gives a relationship between net convergence and net accumulation. There is a further connection. Before we investigate this recall the following: Let (x_i) be a net in a set X based on a directed set W . Let Z be a directed set and let the map $m : Z \rightarrow W$ be a co-final homomorphism. Then the net $(x_{m(k)})$ in X based on Z is said to be a *subnet* of (x_i) . (A subsequence of a sequence is always a subnet of the sequence.)

Theorem 4.14: Let A be an interior algebra, let (a_i) be a net in A and let $b \in A$. Then $(a_i) \rightarrow b$ if and only if there is a subnet (c_k) of (a_i) with $(c_k) \rightarrow b$.

Proof: Let W be the directed set on which (a_i) is based. Suppose $(a_i) \rightarrow b$. Define a directed set Z as follows: Put $Z = \{(i, d) : i \in W \text{ and } d \in N(b) \text{ with } a_i \leq d\}$. Note that $Z \neq \emptyset$ since there is an $i \in W$ whence $(i, 1) \in Z$. Define $\sqsubseteq$ on Z by $(i, d) \sqsubseteq (j, c)$ if and only if $i \sqsubseteq j$ and $d \geq c$. Clearly $\sqsubseteq$ is a pre-order on Z . Let $(i, d), (j, c) \in Z$. There is a $t \in W$ with $i \sqsubseteq t$ and $j \sqsubseteq t$ in W . Also $cd \in N(b)$. Since $(a_i) \rightarrow b$ there is a $s \in W$ with $s \supseteq t$ and $a_s \leq cd$. Then $(s, cd) \in Z$, $(i, d) \sqsubseteq (s, cd)$ and $(j, c) \sqsubseteq (s, cd)$. Thus $Z = \langle Z, \sqsubseteq \rangle$ is a directed set. If $k = (i, d) \in Z$ put $m(k) = i$. Then the map $m : Z \rightarrow W$ is clearly a homomorphism. To see that it is co-final

note that if $i \in W$ then $(i, 1) \in Z$. Hence we have a subnet $(c_k) = (a_{m(k)})$ of (a_i) . Let $d \in N(b)$. There is a $j \in W$. Then there is an $i \supseteq j$ with $a_i \leq d$. Then $(i, d) \in Z$. Let $k = (i, d) \in Z$. Then $c_k = a_i \leq d$. Thus $(c_k) \rightarrow b$. Conversely suppose there is a subnet $(c_k) = (a_{m(k)})$ of (a_i) based on a directed set Z , with $(c_k) \rightarrow b$. Let $d \in N(b)$ and let $j \in W$. There is an $r \in Z$ such that $c_k \leq d$ for all $k \supseteq r$ in Z . Since the map $m : Z \rightarrow W$ is co-final there is a $v \in W$ with $j \sqsubseteq m(v)$ in W . There is a $k \in Z$ with $k \supseteq r$ and $k \supseteq v$. Then $m(k) \supseteq m(v)$ and so $m(k) \supseteq j$. Moreover $a_{m(k)} = c_k \leq d$. Thus $(a_i) \rightarrow b$. $\square$

From Theorem 4.7 and Theorem 4.14 we have:

Corollary 4.15: Let A be an interior algebra, let $R \subseteq A$ and let $b \in A$. Then the following are equivalent:

- (i) $R \rightarrow b$
- (ii) There is a net (a_i) in R with $(a_i) \rightarrow b$.
- (iii) There is a net (a_i) in R with $(a_i) \rightarrow b$. $\square$

Corollary 4.16: Let A be an interior algebra, let (a_i) be a net in A and let (c_k) be a subnet of (a_i) . Then $\text{acc } (c_k) \subseteq \text{acc } (a_i)$.

Proof: Immediate from Theorem 4.14 since any subnet of (c_k) must be a subnet of (a_i) . $\square$

For limits we have the reverse:

Proposition 4.17: Let A be an interior algebra, let (a_i) be a net in A and let (c_k) be a subnet of (a_i) . Then $\lim (a_i) \subseteq \lim (c_k)$.

Proof: Let W and Z be the directed sets on which (a_i) and $(c_k) = (a_{m(k)})$ are respectively based. Suppose $b \in A$ with $(a_i) \rightarrow b$. We must show $(c_k) \rightarrow b$. Let $d \in N(b)$. There is a $j \in W$ such that $a_i \leq d$ for all $i \supseteq j$ in W . But there is a $r \in Z$ such that $m(r) \supseteq j$. Then for all $k \supseteq r$ in Z we have $m(k) \supseteq j$ in W whence $c_k = a_{m(k)} \leq d$. $\square$

In certain cases Theorem 4.7 and Theorem 4.14 can be generalized to sequences.

Theorem 4.18: Let A be an interior algebra, let $R \subseteq A$ and let $b \in A$ such that $N(b)$ is a sequential filter. Then $R \rightarrow b$ if and only if there is sequence (a_n) in R with $(a_n) \rightarrow b$.

Proof: Suppose that $R \rightarrow b$. There is a countable base B for $N(b)$. Let $\{r_n : n < \omega\}$ be an enumeration of B . We define a sequence (a_n) in R as follows. For all $n < \omega$, $r_0 \cdots r_n \in N(b)$ and so there is an $a_n \in R$ with $a_n \leq r_0 \cdots r_n$. Now let $d \in N(b)$. Then for some $k < \omega$ we

have $r_k \leq d$. But then for all $n \geq k$, $a_n \leq r_0 \cdots r_n \leq r_k \leq d$. Thus $(a_n) \rightarrow b$. Conversely if we have a sequence (a_n) in R then $(a_n) \rightarrow b$ by Theorem 4.7. $\square$

Theorem 4.19: Let A be an interior algebra, let (a_n) be a sequence in A and let $b \in A$ such that $N(b)$ is a sequential filter. Then $(a_n) \rightarrow b$ if and only if there is a subsequence (c_k) of (a_n) with $(c_k) \rightarrow b$.

Proof: Let $(a_n) \rightarrow b$. There is a countable base B for $N(b)$. Let $\{r_n : n < \omega\}$ be an enumeration of B . We define (c_k) inductively as follows: $r_0 \in N(b)$ and so there is an $m(0) \geq 0$ such that $a_{m(0)} \leq r_0$. Suppose we have found $m(0) < \dots < m(k)$ for some $k < \omega$ with $a_{m(n)} \leq r_0 \cdots r_n$ for $0 \leq n \leq k$. Now $r_0 \cdots r_{k+1} \in N(b)$ and so there is an $m(k+1) > m(k)$ such that $a_{m(k+1)} \leq r_0 \cdots r_{k+1}$. Now we have a strictly increasing map $m : \omega \rightarrow \omega$ and hence a subsequence $(c_k) = (a_{m(k)})$ of (a_n) . Moreover, for all $k < \omega$, $c_k \leq r_0 \cdots r_k$. Now suppose $d \in N(b)$. Then there is an $n < \omega$ such that $r_n \leq d$. But then for all $k \geq n$ we have $a_k \leq r_0 \cdots r_k \leq r_n \leq d$. Thus $(c_k) \rightarrow b$. The converse follows from Theorem 4.14. $\square$

Corollary 4.20: Let A be an interior algebra, let $R \subseteq A$ and let $b \in A$ such that $N(b)$ is a sequential filter. Then the following are equivalent:

- (i) $R \rightarrow b$
- (ii) There is a sequence (a_n) in R with $(a_n) \rightarrow b$.
- (iii) There is a sequence (a_n) in R with $(a_n) \rightarrow b$. $\square$

Proposition 4.21: Let A be an interior algebra and let $b, c \in A$ with $b > 0$. Consider the following statements:

- (i) $b \leq c$
- (ii) There is a non-zero net (a_i) in $[c]$ with $(a_i) \rightarrow b$.
- (iii) There is a non-zero net (a_i) in $[c]$ with $(a_i) \rightarrow b$.
- (iv) There is a non-zero sequence (a_n) in $[c]$ with $(a_n) \rightarrow b$.
- (v) There is a non-zero sequence (a_n) in $[c]$ with $(a_n) \rightarrow b$.

We always have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (iv). If b is an atom then (i), (ii) and (iii) are equivalent. If $N(b)$ is a sequential filter then (ii) – (v) are all equivalent. In particular if b is an atom and $N(b)$ is sequential, (i) – (v) are all equivalent.

Proof: (i) $\Rightarrow$ (ii): Assume (i). By Proposition 2.19 there is an ultrafilter F in $[c]$ with $F \rightarrow b$ in A . But then $F \subseteq [c] - \{0\}$ and F is a filter base in A . Letting (a_i) be the canonical net of F gives (ii). (ii) $\Rightarrow$ (iii) by Proposition 4.10 (ii), (iii) $\Rightarrow$ (ii) by Theorem 4.14 and clearly (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii). Suppose that b is an atom. Assume (ii). Let $d \in N(b)$. Let W be the directed set on which (a_i) is based. There is a $j \in W$ such that $a_i \leq d$ for all i

$\geq j$ in W , in particular $a_j \leq d$. Thus $0 < a_j \leq cd$. Thus $c \in \text{Sec } N(a)$ and so, since b is an atom, $c \in E(a)$, that is $b \leq c$. Thus if b is an atom (ii) $\Rightarrow$ (i) and so in fact (i), (ii) and (iii) are all equivalent. Suppose now that $N(b)$ is sequential. Assume (ii). Put $S = \{a_i : i \in W\}$ where W is the directed set on which (a_i) is based. By Theorem 4.7, $S \rightarrow b$. By Theorem 4.18 there is a sequence (d_n) in S with $(d_n) \rightarrow b$. Noting that (d_n) is a non-zero sequence in $[c]$ gives (iv). Thus if $N(b)$ is sequential (ii) – (v) are all equivalent. $\square$

Definition 4.22: Let A be an interior algebra and let (a_i) and (b_i) be two nets in A based on the same directed set W . (a_i) is said to *dominate* (b_i) if and only if $a_i \geq b_i$ for all $i \in W$. $\square$

Remark 4.23: Let A be an interior algebra and let (a_i) and (b_i) be two nets in A based on the same directed set W . If (a_i) dominates (b_i) then clearly $\lim (a_i) \subseteq \lim (b_i)$ and $\text{acc } (a_i) \subseteq \text{acc } (b_i)$. If A is an atomic interior algebra, $c \in A$ and (a_i) is a non-zero net in $[c]$, then we can find a net (b_i) in $\text{At } [c]$ which is dominated by (a_i) . It follows that if the interior algebra A in Proposition 4.21 is atomic, then in (ii) – (v) we may replace non-zero nets and non-zero sequences in $[c]$ by nets and sequences in $\text{At } [c]$. $\square$

REFERENCES

1. Blok W.J., *Varieties of Interior Algebras*, Ph.D. Dissertation, University of Amsterdam, 1976
2. Cresswell M.J. & Hughes G.E., *An Introduction to Modal Logic*, Methuen, London, 1968
3. McKinsey J.C.C. & Tarski A., *The Algebra of Topology*, Ann. of Math. 45 (1944) 141-191
4. Naturman C.A. & Rose H., *Ideal Algebras of Interior Algebras*, Bulletin of the Korean Math. Soc., to appear
5. Rasiowa H., *An Algebraic Approach to Non-Classical Logic*, North-Holland Publ. Co., Amsterdam, 1974

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